

# PROMOTION AND ROWMOTION

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**ABSTRACT.** We present an equivariant bijection between two actions—promotion and rowmotion—on order ideals in certain posets. This bijection simultaneously generalizes a result of R. Stanley concerning promotion on the linear extensions of two disjoint chains and recent work of D. Armstrong, C. Stump, and H. Thomas on root posets and noncrossing partitions. We apply this bijection to several classes of posets, obtaining equivariant bijections to various known objects under rotation. We extend the same idea to give an equivariant bijection between alternating sign matrices under rowmotion and under B. Wieland’s gyration. Lastly, we define two actions with related orders on alternating sign matrices and totally symmetric self-complementary plane partitions.

## 1. INTRODUCTION

In his 2009 survey paper on promotion and evacuation [23], R. Stanley gave an equivariant bijection between linear extensions of two disjoint chains  $[n] \oplus [k]$  under promotion and order ideals of the product of two chains  $[n] \times [k]$  under an operation that we call rowmotion. In 2011, D. Armstrong, C. Stump, and H. Thomas gave a uniformly-stated equivariant bijection between noncrossing partitions under Kreweras complementation and nonnesting partitions under rowmotion [1]. In particular, the type  $A$  part of their theorem can be interpreted as an equivariant bijection between linear extensions of  $[2] \times [n]$  under promotion and order ideals of the type  $A$  positive root poset  $\Phi^+(A_n)$  under rowmotion.

We present a new proof of these two theorems by simultaneously generalizing them as a single statement about rc posets—certain posets whose elements and covering relations fit into rows and columns. We give an equivariant bijection between the order ideals of an rc poset  $\mathcal{R}$  under promotion and rowmotion by interpreting promotion as an action on the *columns* of order ideals of  $\mathcal{R}$  and rowmotion as an action on the *rows*. Armed with promotion, we obtain simple equivariant bijections from the order ideals of  $[n] \times [k]$ ,  $J([2] \times [n])$ , positive root posets of types  $A$  and  $B$ , and  $[2] \times [m] \times [n]$  under rowmotion to various known objects under rotation. We also apply this theory to alternating sign matrices (ASMs) and totally symmetric self-complementary plane partitions (TSSCPPs).

The remainder of the paper is structured as follows. In Section 2, we review basic notions about posets, define promotion and rowmotion, and recall the cyclic sieving phenomenon. We briefly summarize the history of rowmotion in Section 3, and build a framework for our results in Sections 4.1 and 4.2 by recalling P. Cameron and D. Fon-Der-Flaass’s toggle group and defining rc posets. We characterize promotion and rowmotion in terms of the toggle group of an rc poset in Section 4.3, which allows us to give an equivariant bijection between promotion and rowmotion for rc posets in Section 5. We apply this equivariant bijection in Section 6 to the product of two chains and the types  $A$  and  $B$  positive root posets, thereby recovering the corresponding results in [23] and [1]. In Section 7 we consider the type  $D$  positive root poset and plane partitions. We turn to ASMs in Section 8 and note that there is an equivariant bijection between ASMs under rowmotion and ASMs under B. Wieland’s gyration. Lastly, we define two actions with related orders on ASMs and TSSCPPs.

## 2. DEFINITIONS

**2.1. Poset Terminology.** A *poset*  $\mathcal{Q}$  is a set with a binary relation “ $\leq$ ” that is reflexive, anti-symmetric, and transitive. If  $q, q' \in \mathcal{Q}$ , we define  $q < q'$  if  $q \leq q'$  and  $q \neq q'$ . We say that  $q'$  covers  $q$  if  $q < q'$  and there does not exist a  $q'' \in \mathcal{Q}$  such that  $q < q'' < q'$ . A poset can be represented by its *Hasse diagram*. To draw the Hasse diagram we represent poset elements as vertices and then draw an edge upward from  $q$  to  $q'$  if and only if  $q'$  covers  $q$ .

**Definition 2.1.** An *order ideal* of  $\mathcal{Q}$  is a set  $J \subseteq \mathcal{Q}$  such that if  $q \in J$  and  $q' \leq q$ , then  $q' \in J$ . We write  $J(\mathcal{Q})$  for the set of all order ideals of  $\mathcal{Q}$ . Recall that  $J(\mathcal{Q})$  forms a distributive lattice under inclusion.

A *partition*  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  is a finite sequence of weakly decreasing positive integers. Partitions have an associated *Ferrers diagram* of boxes, where the number of boxes in the  $i$ th row is equal to the  $i$ th integer in the sequence. Using English notation, we can think of the boxes in a Ferrers diagram as the elements of a poset, where  $x < y$  if the box  $x$  is weakly to the left and above the box  $y$ . For example, in  $\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \hline \end{array}$ , we have  $b < c$  and  $b < e$ , but  $c \not< e$  and  $e \not< c$ . For  $\mu \subseteq \lambda$ , a skew Ferrers diagram  $\lambda/\mu$  consists of the boxes in the Ferrers diagram of  $\lambda$  that are not in  $\mu$ . Let  $[n] = \{1, 2, \dots, n\}$ .

**Definition 2.2.** Let  $\mathcal{Q}$  have  $n$  elements. A *linear extension* of  $\mathcal{Q}$  is a bijection  $\mathcal{L} : \mathcal{Q} \rightarrow [n]$  such that if  $q < q'$ , then  $\mathcal{L}(q) < \mathcal{L}(q')$ . We call linear extensions of a skew Ferrers diagram *Standard Young Tableaux (SYT)*. We write  $\mathcal{L}(\mathcal{Q})$  for the set of all linear extensions of  $\mathcal{Q}$ .

**2.2. Promotion.** In 1972, M.-P. Schützenberger defined promotion as an action on linear extensions [22].

**Definition 2.3.** Let  $\mathcal{L}$  be a linear extension of a poset  $\mathcal{Q}$  and let  $\rho_i$  act on  $\mathcal{L}$  by switching  $i$  and  $i + 1$  if they are not the labels of two elements with a covering relation. We define the *promotion* of  $\mathcal{L}$  to be  $\rho(\mathcal{L}) = \rho_{n-1}\rho_{n-2} \cdots \rho_1(\mathcal{L})$ .

Note that promotion can also be defined using jeu-de-taquin. Since each step of promotion can be reversed,  $\rho$  is a bijection on SYT of a specified shape.

**2.3. Rowmotion.** In 1973, P. Duchet defined an action on hypergraphs [8]. This action was generalized by A. Brouwer and A. Schrijver to an arbitrary poset in [4]. For reasons that will become clear, we will call the action *rowmotion*, denoted  $P$ .

**Definition 2.4.** Let  $\mathcal{Q}$  be a poset, and let  $J \in J(\mathcal{Q})$ . Then  $P(J)$  is the order ideal generated by the minimal elements of  $\mathcal{Q}$  not in  $J$ .

As explained in [7], one motivation for this definition was to study the orbits of the data defining a matroid. For example, working within a Boolean algebra, applying  $P$  to the order ideal generated by the bases of a matroid gives the order ideal generated by the circuits. For more on the history of rowmotion, see Section 3.

**2.4. The Cyclic Sieving Phenomenon.** The Cyclic Sieving Phenomenon was introduced by V. Reiner, D. Stanton, and D. White as a generalization of J. Stembridge’s  $q = -1$  phenomenon [19]. We will use the following definition from their paper, which is one of several equivalents.

**Definition 2.5** (V. Reiner, D. Stanton, D. White). Let  $X$  be a finite set,  $X(q)$  a generating function for  $X$ , and  $C_n$  the cyclic group of order  $n$  acting on  $X$ . Then the triple  $(X, X(q), C_n)$  exhibits the *Cyclic Sieving Phenomenon (CSP)* if for  $c \in C_n$ ,

$$X(\omega(c)) = |\{x \in X : c(x) = x\}|,$$

where  $\omega : C_n \rightarrow \mathbb{C}$  is an isomorphism of  $C_n$  with the  $n$ th roots of unity.

As an example, we have the following theorem. We use the notation  $\binom{[n]}{k}$  for the subsets of  $[n]$  of size  $k$ , and the standard  $q$ -analogues  $[n]_q = \frac{1-q^n}{1-q}$ ,  $[n]!_q = \prod_{i=1}^n [i]_q$ , and  $\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}$ .

**Theorem 2.6** (V. Reiner, D. Stanton, D. White). *Let  $C_n$  act on  $\binom{[n]}{k}$  by the cycle  $(1, 2, \dots, n)$ . Then  $(\binom{[n]}{k}, \binom{n}{k}_q, C_n)$  exhibits the CSP.*

In general, both rowmotion and promotion have orders that are hard to predict.

**Example 2.7.** Promotion has order 7,554,844,752 on SYT of shape  $(8, 6)$ .

M. Haiman and D. Kim classified those SYT with  $n$  boxes on which promotion has order  $n$  or  $2n$ —the generalized staircases, which include rectangles, staircases, and double staircases [11]. In his thesis, B. Rhoades proved a cyclic sieving phenomenon for rectangular SYT under promotion, thereby determining the orbit structure [20]. Similar results for other shapes are limited.

**Theorem 2.8** (B. Rhoades). *Let  $\lambda = (n, n, \dots, n)$  be a rectangular partition of  $n \cdot m$ , and let  $\text{SYT}(\lambda)$  be the set of SYT of shape  $\lambda$ . Let  $C_n$  act on  $\text{SYT}(\lambda)$  by promotion and let*

$$f^\lambda(q) = \frac{[n \cdot m]!_q}{\prod_{i,j} [h_{i,j}]_q}$$

*be the  $q$ -analogue of the hook-length formula for  $\text{SYT}(\lambda)$ . Then  $(\text{SYT}(\lambda), f^\lambda(q), C_{n \cdot m})$  exhibits the CSP.*

### 3. HISTORY

In this section, we recall known results for rowmotion acting on  $[n] \times [k]$  and positive root posets. We phrase these results as equivariant bijections between linear extensions of a poset under promotion and order ideals of a different poset under rowmotion.

**3.1. Products of Two Chains.** The problem of determining the order of rowmotion on the product of two chains was proposed in a 1974 paper by A. Brouwer and A. Schrijver [4]. After showing that the order of rowmotion on a Boolean algebra failed to adhere to a conjectural pattern for  $n > 4$ , the two proved the following theorem:

**Theorem 3.1** (A. Brouwer, A. Schrijver).  *$J([n] \times [k])$  under  $P$  has order  $n + k$ .*

D. Fon-der-Flaass used a clever combinatorial model to refine this result in 1992 in [10]:

**Theorem 3.2** (D. Fon-der-Flaass). *The length of any orbit of  $J([n] \times [k])$  under  $P$  is  $(n + k)/d$  for some  $d$  dividing both  $n$  and  $k$ . Any number of this form is the length of some orbit.*

In his 2009 survey paper [23], R. Stanley noted that there was an equivariant bijection between promotion and rowmotion. This completely resolved the original problem.

**Theorem 3.3** (R. Stanley). *There is an equivariant bijection between  $\mathcal{L}([n] \oplus [k])$  under  $\rho$  and  $J([n] \times [k])$  under  $P$ .*

Note that  $\mathcal{L}([n] \oplus [k])$  can be thought of as skew SYT of shape  $(n) \oplus (k)$ . It is also in equivariant bijection with the set  $\binom{[n+k]}{k}$  under the cycle  $(1, 2, \dots, n+k)$ , so that Theorem 2.6 applies. Figure 1 illustrates this theorem for the case  $n = k = 2$ .

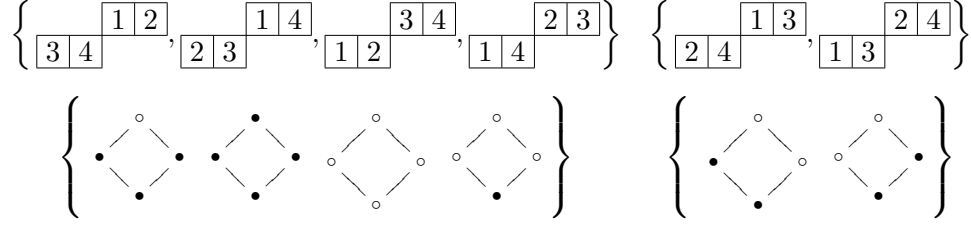


FIGURE 1. The two orbits of  $\mathcal{L}([2] \oplus [2])$  under  $\rho$  and the two orbits of  $J([2] \times [2])$  under  $P$ .

**3.2. Positive Root Posets.** Let  $W$  be a crystallographic Weyl group for a root system  $\Phi(W)$ . We shall denote the positive root poset of type  $W$  as  $\Phi^+(W)$ , where if  $\alpha, \beta \in \Phi^+(W)$ , then  $\alpha \leq \beta$  if  $\beta - \alpha$  is a nonnegative sum of positive roots.

The set of positive roots for types  $A_n$ ,  $B_n$ ,  $C_n$ , and  $D_n$  are:

- $\Phi^+(A_n) = \{e_i - e_j | 1 \leq i < j \leq n + 1\}$
- $\Phi^+(B_n) = \{e_i \pm e_j | 1 \leq i < j \leq n\} \cup \{e_i | 1 \leq i \leq n\}$
- $\Phi^+(C_n) = \{e_i \pm e_j | 1 \leq i < j \leq n\} \cup \{2e_i | 1 \leq i \leq n\}$
- $\Phi^+(D_n) = \{e_i - e_{i+1} | 1 \leq i \leq n - 1\} \cup \{e_{n-1} + e_n\}$ .

$\Phi^+(C_n)$  is isomorphic to  $\Phi^+(B_n)$ . Figure 2 gives the positive root posets of  $A_3$ ,  $B_3$ ,  $C_3$ , and  $D_4$  as examples.

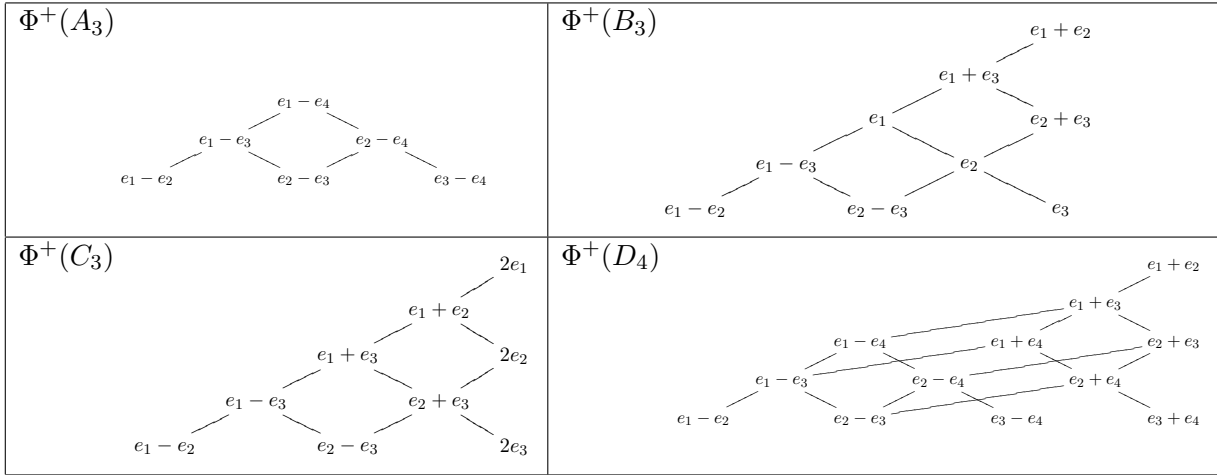


FIGURE 2. The positive root posets  $A_3$ ,  $B_3$ ,  $C_3$ , and  $D_4$ .

Let  $h$  be the Coxeter number for  $W$ , and let  $d_1, d_2, \dots, d_n$  be the degrees of  $W$ . Recall that  $Cat(W, q) = \prod_{i=1}^n \frac{[h+d_i]_q}{[d_i]_q}$ , so that  $Cat(W, 1) = |J(\Phi^+(W))|$ .

In 2007, D. Panyushev considered applying rowmotion to order ideals of positive root posets  $\Phi^+(W)$  [17], which are well-known to be in bijection with non-nesting partitions of type  $W$ . He conjectured several properties of rowmotion, the most relevant to our story being:

**Conjecture 3.4** (D. Panyushev). *The order of  $P$  on  $J(\Phi^+(A_n))$  is  $2(n+1) = 2h$ , and the order is  $h$  for all other types.*

D. Bessis and V. Reiner then made the stronger conjecture that there was a CSP [3].

**Conjecture 3.5** (D. Bessis, V. Reiner). *Let  $C_{2h}$  act on  $J(\Phi^+(W))$  by  $P$ . Then  $(J(\Phi^+(W)), Cat(W, q), C_{2h})$  exhibits the CSP.*

These two conjectures were recently proved by D. Armstrong, C. Stump, and H. Thomas in [1], in which they constructed equivariant bijections from rowmotion on order ideals of a positive root poset to rotation of noncrossing partitions of the corresponding type. The noncrossing partitions under rotation are known to have the order conjectured by D. Panyushev and to exhibit the CSP, from which the results follow.

**Theorem 3.6** (D. Armstrong, C. Stump, and H. Thomas). *There is a uniformly-stated equivariant bijection between non-nesting partitions under rowmotion and noncrossing partitions under Kreweras complementation.*

In any type, rotation of a noncrossing matching is in equivariant bijection with Kreweras complementation on noncrossing partitions. Furthermore, D. White constructed an equivariant bijection between type  $A_n$  noncrossing matchings under rotation and rectangular SYT with two rows under promotion.

**Theorem 3.7** (D. White). *An equivariant bijection between type  $A_n$  noncrossing matchings under rotation and SYT of shape  $(n+1, n+1)$  under promotion is given by placing  $i$  in the first row when it is the smaller of the two numbers in its matching.*

In analogy with Theorem 3.3, we can restate the type  $A_n$  result of Theorem 3.7 in the language of promotion.

**Theorem 3.8.** *There is an equivariant bijection between  $\mathcal{L}([2] \times [n+1])$  under  $\rho$  and  $J(\Phi^+(A_n))$  under  $P$ .*

Note that  $\mathcal{L}([2] \times [n+1])$  are SYT of shape  $(n+1, n+1)$ . Figure 3 illustrates this theorem for  $n = 2$ .

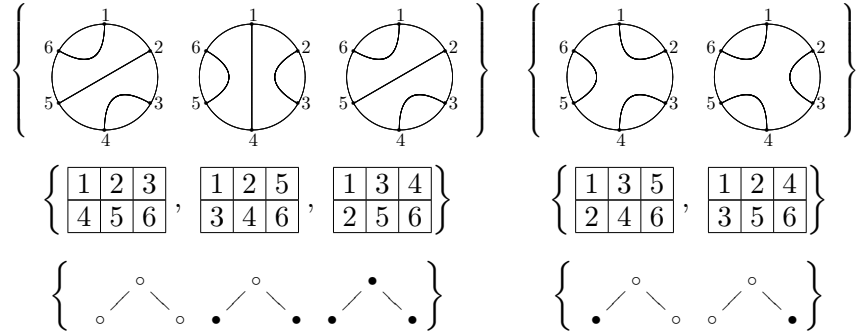


FIGURE 3. The two orbits of noncrossing matchings on six points under rotation, the two orbits of SYT of shape  $(3, 3)$  under  $\rho$  and the two orbits of  $J(\Phi^+(A_2))$  under  $P$ .

Our main theorem, Theorem 5.2, gives an equivariant bijection between promotion and rowmotion on the order ideals of any poset with rows and columns (in a sense we will make precise in Section 4.2). In particular, the result holds for *all* skew SYT with at most two rows, and so we obtain Theorems 3.3 and 3.8 as special cases.

#### 4. MACHINERY

In this section, we develop the machinery needed to prove our main theorem. We first recall P. Cameron and D. Fon-der-Flaass's permutation group on the order ideals of a poset, which we call the toggle group. We then define rc posets, interpret promotion and rowmotion as elements in the toggle group of an rc poset, and show that promotion and rowmotion are conjugate elements in

these toggle groups. The following lemma then specifies an equivariant bijection between the order ideals of rc posets under promotion and the order ideals of rc posets under rowmotion.

**Lemma 4.1.** *Let  $G$  be a group acting on a set  $X$ , and let  $g_1$  and  $g_2 = gg_1g^{-1}$  be conjugate elements in  $G$ . Then  $x \rightarrow gx$  gives an equivariant bijection between  $X$  under  $\langle g_1 \rangle$  and  $X$  under  $\langle g_2 \rangle$ .*

**4.1. The Toggle Group.** Let  $\mathcal{Q}$  be a poset and let  $J(\mathcal{Q})$  be its set of order ideals. In [5], P. Cameron and D. Fon-der-Flaass defined a group acting on  $J(\mathcal{Q})$ .

**Definition 4.2** (P. Cameron and D. Fon-der-Flaass). For each  $q \in \mathcal{Q}$ , define  $t_q : J(\mathcal{Q}) \rightarrow J(\mathcal{Q})$  to act by toggling  $q$  if possible. That is, if  $J \in J(\mathcal{Q})$ ,

$$t_q(J) = \begin{cases} J \cup \{q\} & \text{if } q \notin J \text{ and if } q' < q \text{ then } q' \in J, \\ J - q & \text{if } q \in J \text{ and if } q' > q \text{ then } q' \notin J, \\ J & \text{otherwise.} \end{cases}$$

**Definition 4.3** (P. Cameron and D. Fon-der-Flaass). The toggle group  $T(\mathcal{Q})$  of a poset  $\mathcal{Q}$  is the subgroup of the permutation group  $\mathfrak{S}_{J(\mathcal{Q})}$  generated by  $\{t_q\}_{q \in \mathcal{Q}}$ .

Note that  $T(\mathcal{Q})$  has the following obvious relations (which do not constitute a full presentation):

- (1)  $t_q^2 = 1$ , and
- (2)  $(t_q t_{q'})^2 = 1$  if  $q$  and  $q'$  do not have a covering relation.

We can use the toggle group to provide an alternative characterization of rowmotion as an element of  $T(\mathcal{Q})$ .

**Theorem 4.4** (P. Cameron and D. Fon-der-Flaass). *Fix a linear extension  $\mathcal{L}$  of  $\mathcal{Q}$ . Then*

$$t_{\mathcal{L}^{-1}(1)} t_{\mathcal{L}^{-1}(2)} \cdots t_{\mathcal{L}^{-1}(n)}$$

*acts as  $P$ .*

**4.2. Rowed-and-Columned Posets.** We now define rc posets—certain posets whose elements fit into rows and columns with covering relations allowed only between diagonally adjacent elements. We will interpret promotion as an action that toggles the columns of order ideals of rc posets, and rowmotion as an action that toggles the rows.

**Definition 4.5.** Let  $\Pi \subset \mathbb{R}^2$  be the set of points in the integer span of  $(2, 0)$  and  $(1, 1)$ . A *rowed-and-columned (rc) poset*  $\mathcal{R}$  is a finite poset together with a map  $\pi : \mathcal{R} \rightarrow \Pi$ , where if  $r_1, r_2 \in \mathcal{R}$ ,  $r_1$  covers  $r_2$ , and  $\pi(r_1) = (i, j)$ , then  $\pi(r_2) = (i + 1, j - 1)$  or  $\pi(r_2) = (i - 1, j - 1)$ . For  $r \in \mathcal{R}$ , we call  $\pi(r) \in \Pi$  the *position* of  $r$ .

Let the *height*  $h$  of an rc poset be the maximum number of elements in a single position  $(i, j)$ . The  $j$ th row of an rc poset  $\mathcal{R}$  is the set of elements of  $\mathcal{R}$  in positions  $\{(i, j)\}_i$ . The  $i$ th column of an rc poset is the set of elements of  $\mathcal{R}$  in positions  $\{(i, j)\}_j$ . Let  $n$  denote the maximal non-empty row and  $k$  the maximal non-empty column. For example, see Figure 4.

By definition, an element is in a row and column of the same parity. This is the key to our proof of Theorem 5.2.

**Example 4.6.** Other examples of rc posets of height 1 are:

- (1)  $[n] \times [k]$  (take  $\pi((i, j)) = (i - j, i + j)$ ),
- (2)  $\Phi^+(A_n)$  (take  $\pi(e_i - e_j) = (i + j, j - i)$ ), and
- (3)  $\Phi^+(B_n) \cong \Phi^+(C_n)$  (for  $\Phi^+(B_n)$ , take  $\pi(e_i - e_j) = (i + j, j - i)$ ,  $\pi(e_i) = (n + 1 + i, n + 1 - i)$ , and for  $i < j$  let  $\pi(e_i + e_j) = (2n + 2 - (j - i), 2n + 2 - (i + j))$ ).

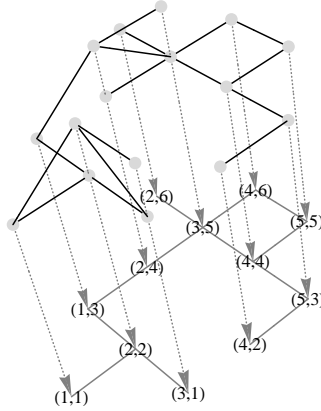


FIGURE 4. This picture represents an rc poset with height  $h = 2$ ,  $k = 5$  columns, and  $n = 6$  rows. When there are two elements with the same position, the second element is raised; the position is indicated by a dotted arrow down. Covering relations are drawn with solid black lines and are projected down as solid gray lines.

We will consider certain posets of height 1 in Section 6. We remark here that  $\Phi^+(D_n)$  can be drawn as an rc poset of height 2 (see Section 7).

On rc posets  $\mathcal{R}$ , promotion can be defined as an action that scans across the *columns* of  $\mathcal{R}$ , and rowmotion as an action that scans down the *rows*. By the commutation relations of the toggle group, the order that we take the elements within a row or column does not matter.

**Definition 4.7.** If  $\mathcal{R}$  is an rc poset, let  $r_i = \prod t_q$ , where the product is over all elements in row  $i$  and let  $c_i = \prod t_q$ , where the product is over all elements in column  $i$ .

Then, since no elements within a row or column of an rc poset share a covering relation, the following relations hold:

- (1)  $r_i^2 = c_i^2 = 1$ , and
- (2) if  $|i - j| > 1$ ,  $(r_i r_j)^2 = (c_i c_j)^2 = 1$ .

**4.3. Promotion and Rowmotion in the Toggle Group.** We interpret promotion and rowmotion as elements of the toggle group of an rc poset with  $n$  rows and  $k$  columns. Without loss of generality—purely for ease of notation—we assume that the rc poset is translated into the first quadrant so that the rows are labeled from 1 to  $n$  and the columns from 1 to  $k$ .

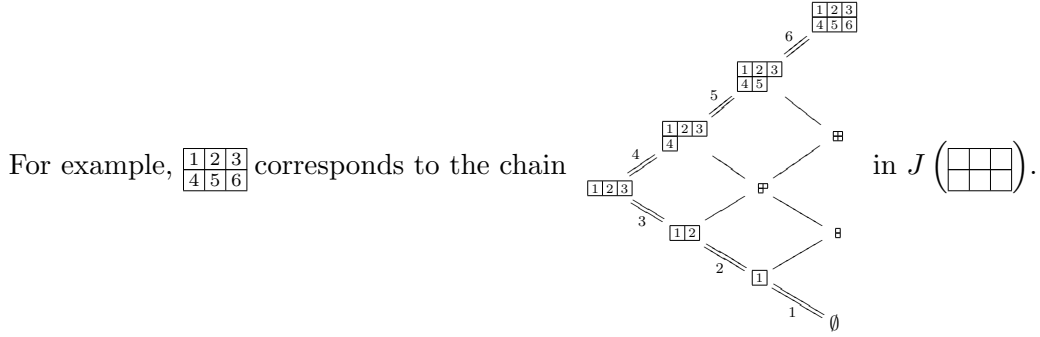
**Definition 4.8.** (1) Given  $\nu \in \mathfrak{S}_k$  let  $\rho_\nu = \prod_{i=1}^k c_{\nu(i)} = c_{\nu(1)} \cdot c_{\nu(2)} \cdots c_{\nu(k)}$ .  
(2) Likewise, given  $\omega \in \mathfrak{S}_n$  let  $P_\omega = \prod_{i=1}^n r_{\omega(i)}$ .

We now specify the element of the toggle group that we will take to act as rowmotion.

**Corollary 4.9.** On rc posets,  $P_{12\dots n}$  acts as  $P$ .

*Proof.* This follows immediately from Theorem 4.4. □

Interpreting promotion as an element of the toggle group takes slightly more work. Let  $\mathcal{Q}$  be a Ferrers diagram. Following R. Stanley in [23], we define promotion using the order ideals  $J(\mathcal{Q})$ . Linear extensions  $\mathcal{L}$  can be interpreted as maximal chains  $\emptyset = J_0 \subset J_1 \cdots \subset J_n = \mathcal{Q}$  in  $J(\mathcal{Q})$  by taking  $\mathcal{L}(q) = i$  if  $q \in J_{i+1} - J_i$ .



The promotion of  $\lambda = (\emptyset = J_0 \subset J_1 \cdots \subset J_n = \mathcal{Q})$  is  $\tau_{n-1} \cdots \tau_1 \lambda$ , where  $\tau_i$  acts on a chain by switching  $J_i$  to the other order ideal in  $\{J_{i-1}, J_{i+1}\}$ , if one exists. Figure 5 illustrates promotion on the maximal chains.

When  $\mathcal{Q}$  is a Ferrers diagram with at most two rows, we can draw the Hasse diagram of  $J(\mathcal{Q})$  as a planar poset. The  $i$ th step of a maximal chain in  $J(\mathcal{Q})$  is taken to be northwest if  $i$  is in the first row, and northeast otherwise. We take advantage of this planarity with the following definition.

**Definition 4.10.** If  $\mathcal{Q}$  is a skew Ferrers diagram with at most two rows, define  $J(\mathcal{Q})^o$  to be the rc poset with elements the boxes of  $J(\mathcal{Q})$  and covering relations between two elements when their corresponding boxes are adjacent.

Therefore, when  $\mathcal{Q}$  has at most two rows, a maximal chain in  $J(\mathcal{Q})$  traces out an order ideal—defined by the boxes to the right of the maximal chain—in  $J(\mathcal{Q})^o$ .

**Definition 4.11.** More generally, we define the *boundary path* of an order ideal of a connected rc poset of height 1 to be the path that separates the order ideal from the rest of the poset. We can encode boundary paths as binary words by writing a 1 for a northeast step and a 0 for a southeast step.

**Example 4.12.** The Hasse diagram of  $J\left(\begin{smallmatrix} & & \\ & & \end{smallmatrix}\right)$  (with the boxes of the Hasse diagram labeled by

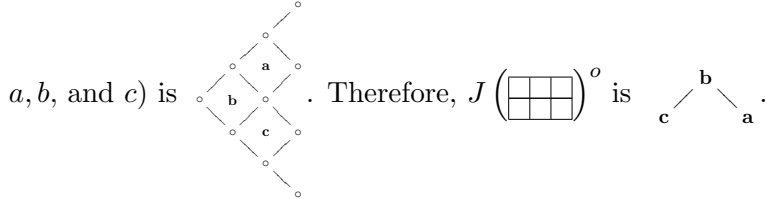


Figure 5 illustrates the bijection from SYT of shape  $(3, 3)$  to order ideals of  $\Phi^+(A_2)$ .

**Theorem 4.13.** When  $\mathcal{R} = J(\mathcal{Q})^o$  is an rc poset, then  $\rho_{k \dots 21}$  acts as promotion on SYT of shape  $\mathcal{Q}$ .

*Proof.* This follows from the characterization of promotion as an action on maximal chains in  $J(\mathcal{Q})$ .  $\square$

We can reverse this process as follows.

**Proposition 4.14.** Draw the boundary path of an order ideal of an rc poset  $\mathcal{R} = J(\mathcal{Q})^o$ . An equivariant bijection between order ideals of  $\mathcal{R}$  under  $\rho$  and SYT of shape  $\mathcal{Q}$  under  $\rho$  is given by placing  $i$  in the top row of  $\mathcal{Q}$  if the  $i$ th step of the boundary path is northeast, and putting  $i$  in the bottom row otherwise.

When we start with the poset  $\Phi^+(A_n)$  under  $\rho$  and map to SYT as above, we can apply Theorem 3.7 to obtain noncrossing matchings under rotation. In this language,  $i$  is the smaller number in its partition if the  $i$ th step of the boundary path is northeast.



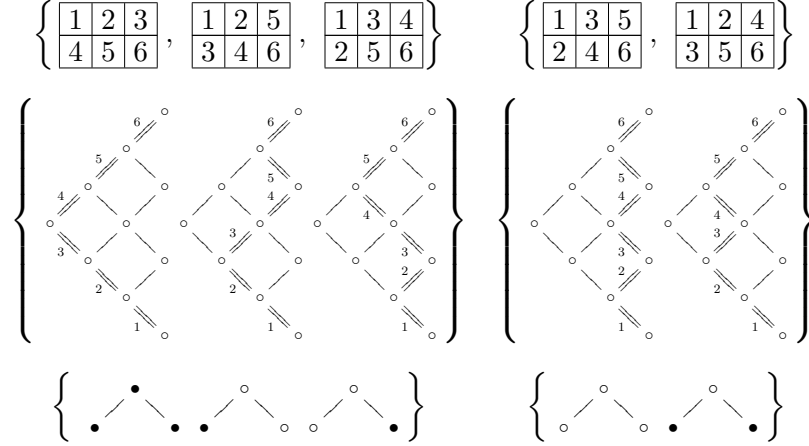


FIGURE 5. The two orbits of SYT of shape  $(3, 3)$  under promotion, the same two orbits using the maximal chain interpretation, and the same two orbits using the order ideal interpretation.

We apply this idea of boundary paths under  $\rho$  to noncrossing objects under rotation in Section 6, and generalize it in Section 7. In Sections 7 and 8, we conjecture that there is a further generalization to the type  $D_n$  positive root poset, plane partitions, the ASM poset, and the TSSCPP poset.

## 5. THE CONJUGACY OF PROMOTION AND ROWMOTION

We now prove that promotion and rowmotion are conjugate elements in the toggle group of an rc poset and then spend the rest of the paper applying this theorem to specific rc posets.

**Lemma 5.1** ([12]). *Let  $G$  be the group generated by  $g_1, \dots, g_n$  with  $g_i^2 = 1$  and  $(g_i g_j)^2 = 1$  if  $|i - j| > 1$ . Then for any  $\omega, \nu \in \mathfrak{S}_n$ ,  $\prod_i g_{\omega(i)}$  and  $\prod_i g_{\nu(i)}$  are conjugate.*

**Theorem 5.2.** *For any rc poset  $\mathcal{R}$  and any  $\omega \in \mathfrak{S}_n$  and  $\nu \in \mathfrak{S}_k$ , there is an equivariant bijection between  $J(\mathcal{R})$  under  $P_\omega$  and  $J(\mathcal{R})$  under  $\rho_\nu$ .*

*Proof.* Since the row toggles  $r_i$  satisfy the conditions of Lemma 5.1, for any rc poset  $\mathcal{R}$  and any  $\omega, \nu \in \mathfrak{S}_n$ , there is an equivariant bijection between  $J(\mathcal{R})$  under  $P_\omega$  and  $J(\mathcal{R})$  under  $P_\nu$ . Similarly, since the column toggles  $c_i$  satisfy the conditions of Lemmas 5.1 and 4.1, for any  $\omega, \nu \in \mathfrak{S}_k$ , there is an equivariant bijection between  $J(\mathcal{R})$  under  $\rho_\omega$  and  $J(\mathcal{R})$  under  $\rho_\nu$ .

Therefore, we may restrict to considering only  $P_{135\dots 246\dots}$  and  $\rho_{135\dots 246\dots}$ . But since all  $t_p$  with  $p$  in an odd (resp. even) column or row commute with one another, and since elements in an odd (resp. even) row are also necessarily in an odd (resp. even) column, we conclude that  $P_{135\dots 246\dots}$  is equal to  $\rho_{135\dots 246\dots}$ .  $\square$

We may ask for an explicit equivariant bijection from rowmotion  $P_{12\dots n}$  to promotion  $\rho_{k\dots 21}$ . It is more convenient to go from  $P^{-1} = P_{n\dots 21}$  to  $\rho_{k\dots 21}$ . To this end, define the  $j$ th diagonal of an rc poset to be the set of elements in positions  $\{(2(j-1) + i, i)\}_i$  that lie in  $\mathcal{R}$ . Let  $m$  be the maximal non-empty diagonal.

**Definition 5.3.** If  $\mathcal{R}$  is an rc poset, let  $d_j = \prod t_q$ , where the product is over all elements in diagonal  $j$ . The order within a diagonal does matter, and we specify the order of the elements to be (from left to right) from smallest row to largest row.

**Theorem 5.4.** *An equivariant bijection from  $J(\mathcal{R})$  under  $P_{n\dots 21}$  and  $J(\mathcal{R})$  under  $\rho_{k\dots 21}$  is given by acting on an order ideal by  $D = \prod_{i=m}^2 \prod_{j=i}^m d_j$ .*

*Proof.* We show that  $DP_{n\dots 21}D^{-1} = \rho_{k\dots 21}$ . Using the commutation relations,

$$P_{n\dots 21}d_m^{-1}d_{m-1}^{-1}\cdots d_2^{-1} = d_1,$$

so that

$$P_{n\dots 21}D^{-1} = d_1 \prod_{i=3}^m \prod_{j=m}^i d_j^{-1}.$$

Since no elements of  $\prod_{i=3}^m \prod_{j=m}^i d_j^{-1}$  are in the first or second diagonal, they may commute through  $d_1$ . Therefore,

$$DP_{n\dots 21}D^{-1} = \left( \prod_{i=m}^2 \prod_{j=i}^m d_j \right) \left( \prod_{i=3}^m \prod_{j=m}^i d_j^{-1} \right) d_1 = d_m d_{m-1} \cdots d_1,$$

which is easy to show commutes to equal  $\rho_{k\dots 21}$ .  $\square$

## 6. RC POSETS OF HEIGHT $h = 1$

We apply Theorem 5.2 to the following rc posets of height one:  $[n] \times [k]$ ,  $J([2] \times [n-1])$ ,  $\Phi^+(A_n)$ , and  $\Phi^+(B_n) \cong \Phi^+(C_n)$ .

6.1.  $[n] \times [k]$ . As a corollary of Theorem 5.2 we obtain a new proof of Theorem 3.3.

*Proof of Theorem 3.3.* By the reasoning in Section 4.3,  $\mathcal{L}([n] \oplus [k])$  under  $\rho$  is in equivariant bijection with  $J([n] \times [k])$  under  $\rho$ . The result then follows from Theorem 5.2.  $\square$

Since  $\mathcal{L}([n] \oplus [k])$  under  $\rho$  is in bijection with  $\binom{[n+k]}{k}$  under the cycle  $(1, 2, \dots, n+k)$ , we can restate the theorem using the map from Proposition 4.14.

**Theorem 6.1.** *There is an equivariant bijection between  $J \in J([n] \times [k])$  under  $P$  and binary words of the form  $w(J)$  under rotation, where  $w(J) = w_1 w_2 \dots w_{n+k}$  is a binary word of length  $n+k$  with  $n$  1's.*

The bijection is given by using our bijection from  $J([n] \times [k])$  under  $P$  to  $J([n] \times [k])$  under  $\rho$ , and then setting  $w_i$  to 1 if the  $i$ th step of the boundary path is northeast, and to 0 otherwise.

A CSP follows immediately from Theorem 2.6.

6.2.  $J([2] \times [n-1])$ . Observe that  $J([2] \times [n-1])$  can be embedded as the left half of  $[n] \times [n]$ . It is not hard to see that the map from Proposition 4.14 can be adapted.

**Theorem 6.2.** *There is an equivariant bijection between  $J \in J(J([2] \times [n-1]))$  under  $P$  and binary words of the form  $w(J)(1 - w(J))$  under rotation, where  $w(J) = w_1 w_2 \dots w_n$  is any binary word of length  $n$ , and  $1 - w(J)$  is the word of length  $n$  whose  $i$ th letter equals  $1 - w_i$ .*

Again, we first use our bijection from  $J(J([2] \times [n-1]))$  under  $P$  to  $J(J([2] \times [n-1]))$  under  $\rho$ , and then set  $w_i$  equal to 1 if the  $i$ th step of the boundary path is northeast, and 0 otherwise. This theorem is illustrated for the case  $n = 3$  in Figure 6.

The set  $X$  of binary words of the form  $w(1 - w)$ , where  $w$  is a binary word of length  $n$ , exhibits the CSP under rotation with the polynomial  $\prod_{i=1}^n [2]_{q^i}$ .

**Corollary 6.3.** *Let  $C_{2n}$  act on  $J(J([2] \times [n-1]))$  by  $P$ . Then  $(J(J([2] \times [n-1])), \prod_{i=1}^n [2]_{q^i}, C_{2n})$  exhibits the CSP.*

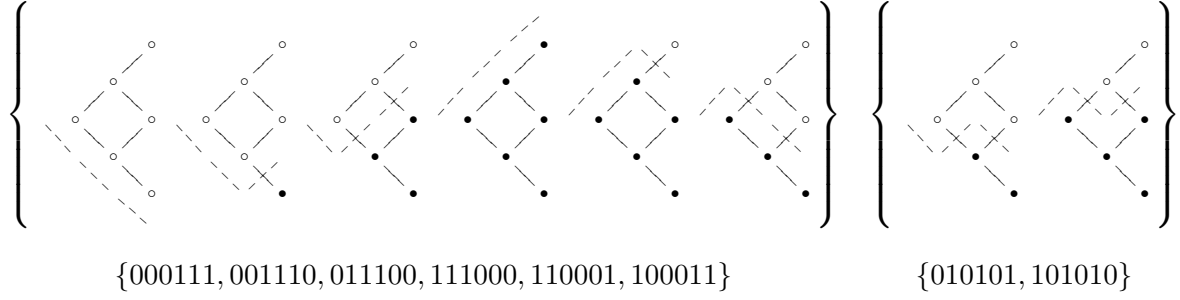


FIGURE 6. There are two orbits of  $J(J([2] \times [2]))$  under  $\rho$  (the dashed lines are the boundary paths corresponding to the order ideals) and two orbits of binary words of length 6 of the form  $w(1-w)$  under rotation (obtained from the boundary paths).

6.3.  $\Phi^+(A_n)$ . Using D. White's equivariant bijection between  $\mathcal{L}([2] \times [n+1])$  and noncrossing matchings, we obtain the type  $A_n$  case of Theorem 3.6.

*Proof of Theorem 3.8.* By the reasoning in Section 4.3,  $\mathcal{L}([2] \times [n+1])$  under  $\rho$  is in equivariant bijection with  $J(\Phi^+(A_n))$  under  $\rho$ . The result then follows from Theorem 5.2.  $\square$

6.4.  $\Phi^+(B_n)$ . The type  $B_n$  case of Theorem 3.6 also follows from a modification of the map in Proposition 4.14, since  $B_n$  noncrossing matchings are just the half-turn symmetric  $A_{2n-1}$  matchings.

**Corollary 6.4.** *There is an equivariant bijection between type  $B_n$  noncrossing matchings under rotation and  $J(\Phi^+(B_n))$  under  $P$ .*

Figure 7 illustrates this theorem for  $n = 2$ .

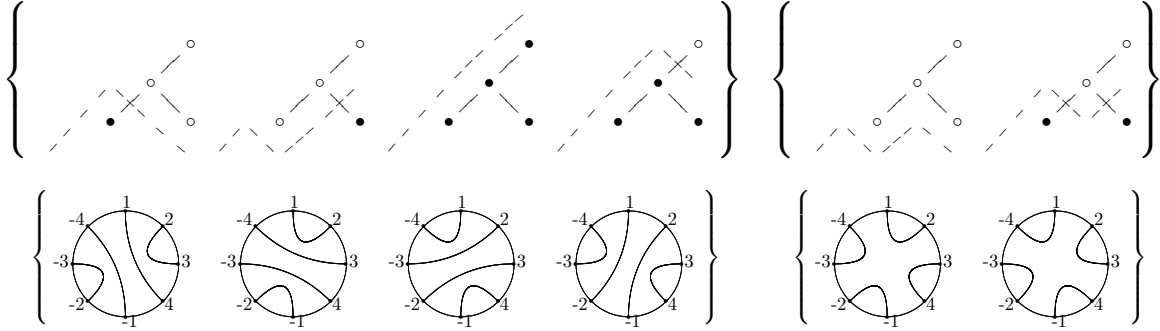


FIGURE 7. The two orbits of  $J(\Phi^+(B_2))$  under  $\rho$  (the dashed lines are the boundary paths corresponding to the order ideals) and the two orbits of type  $B_2$  noncrossing matchings under rotation (obtained from the boundary path by taking  $i$  to be the smaller element of its block if the  $i$ th step was northeast).

## 7. RC POSETS OF HEIGHT $h > 1$

We now apply Theorem 5.2 to the following rc posets of height greater than one:  $\Phi^+(D_n)$  and  $[\ell] \times [m] \times [n]$ .

7.1.  $\Phi^+(D_n)$ . The poset  $\Phi^+(D_n)$  is a copy of  $\Phi^+(A_{n-1})$  joined with  $J([2] \times [n-2])$  (see Figure 2). We choose to draw  $\Phi^+(D_n)$  as an rc poset of height 2 by letting the elements  $e_i - e_n$  and  $e_i + e_n$  occupy the same positions. Then—ignoring elements and edges between elements in same position— $\Phi^+(D_n)$  looks exactly like  $\Phi^+(B_n)$ . So for our position map, for  $i < j$  we take  $\pi(e_i - e_j) = (i+j, j-i)$  and  $\pi(e_i + e_j) = (2n - (j - i), 2n - (i + j))$ . For example,  $D_4$  is drawn this way in Figure 8.

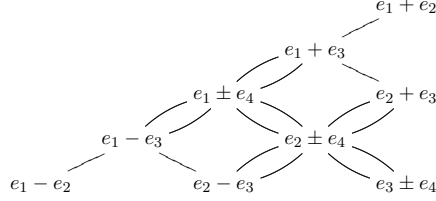


FIGURE 8.  $\Phi^+(D_4)$  drawn as an rc poset of height 2.

As a corollary of Theorem 5.2 we obtain the following.

**Corollary 7.1.** *There is an equivariant bijection between  $J(\Phi^+(D_n))$  under  $P$  and  $J(\Phi^+(D_n))$  under  $\rho$ .*

Recall that type  $D_n$  noncrossing matchings are defined to be half-turn symmetric noncrossing matchings satisfying a certain parity condition on  $4n - 4$  points around a large circle and 4 points around a smaller interior circle [2]. Rotation on these is defined by rotating the inner and outer circles in opposite directions.

Let  $J \in J(\Phi^+(D_n))$  have the property that  $e_i + e_n \in J$  if and only if  $e_i - e_n \in J$ . Then  $\rho(J)$  also has this property, so that the entire orbit is mirrored by the corresponding orbit in  $J(\Phi^+(B_n))$  obtained by identifying  $e_i + e_n$  and  $e_i - e_n$ . Thus, such order ideals are in bijection with those type  $D_n$  noncrossing matchings that have no matchings between the outer vertices and the four inner vertices. The general case seems more complicated (though it was solved in [1]), and we leave it as a conjecture.

**Conjecture 7.2.** *The map in Proposition 4.14 can be extended to equivariantly map elements of  $J(\Phi^+(D_n))$  under  $\rho$  to type  $D_n$  noncrossing matchings under rotation.*

7.2. **Plane Partitions.** In this section, consider the order ideals of the product of three chains—that is, plane partitions—under rowmotion. We draw  $[\ell] \times [m] \times [n]$  as an rc poset of height  $\ell$  to generalize the approach in Proposition 4.14, simplifying proofs due to P. Cameron and D. Fon-der-Flaass and D. B. Rush and X. Shi.

We have already dealt with the  $[1] \times [m] \times [n]$  case in Theorem 3.3. In [5], P. Cameron and D. Fon-der-Flaass prove the following theorem for  $[2] \times [m] \times [n]$ .

**Theorem 7.3** (P. Cameron, D. Fon-der-Flaass). *The order of  $P$  on  $J([2] \times [m] \times [n])$  is  $m + n + 1$ .*

They proved this theorem by constructing a bijection between entire orbits of  $J([2] \times [m] \times [n])$  under a conjugate of rowmotion and orbits of certain words under an action  $\psi$  whose order is more easily analyzed. By interpreting plane partitions as rc posets, we will simplify this proof by giving a bijection from plane partitions under promotion to these words under  $\psi$ .

A word containing parentheses is called *balanced* if the number of left parentheses is always greater than or equal to the number of right parentheses.

**Definition 7.4** (P. Cameron, D. Fon-der-Flaass). Let  $\beta_{m,n}$  be all balanced words of length  $m+n+1$  on the alphabet  $\{ (, ), \bullet, \boxed{ } \}$  with  $m$  left parentheses and  $m$  right parentheses (including those in a  $\boxed{ } ($  symbol).

Define an action  $\psi$  on  $\beta_{m,n}$  as follows:

- (1)  $\psi[\bullet A_1] = A_1 \bullet$ ,
- (2)  $\psi[(A_1)A_2] = A_1(A_2)$ ,
- (3)  $\psi[(A_1)(A_2)(\dots)(A_k)A_{k+1}] = A_1(A_2)(\dots)(A_k)(A_{k+1})$ ,

where each of  $A_1, A_2, \dots, A_{k+1}$  is a balanced subword.

**Theorem 7.5** (P. Cameron, D. Fon-der-Flaass). *There is an equivariant bijection between  $J([2] \times [m] \times [n])$  under  $P$  and  $\beta_{m,n}$  under  $\psi$ .*

Recall that the generating function for plane partitions inside an  $[\ell] \times [m] \times [n]$  box is given by the  $q$ -ification of the MacMahon box formula [15].

$$M_{\ell,m,n}(q) = \prod_{1 \leq i \leq \ell, 1 \leq j \leq m, 1 \leq k \leq n} \frac{[i+j+k-1]_q}{[i+j+k-2]_q}$$

In general,  $(J([\ell] \times [m] \times [n]), M_{\ell,m,n}(q), C_{\ell+m+n-1})$  does not exhibit the CSP, where the cyclic group acts by rowmotion. For  $[3] \times [3] \times [3]$ , the polynomial fails to return valid numbers of orbits, while for  $[4] \times [4] \times [4]$ , K. Dilks computed that there exist orbits of size  $33 = 3 \cdot (4 + 4 + 4 - 1)$ .

V. Reiner originally conjectured that  $(J([2] \times [m] \times [n]), M_{2,m,n}(q), C_{m+n+1})$  exhibited the CSP. D. B. Rush and X. Shi recently proved this using P. Cameron and D. Fon-der-Flaass's equivariant bijection to  $\beta_{m,n}$  [21]. Their theorem, which we obtain as a corollary, was the inspiration for our bijection to noncrossing partitions in Theorem 7.8.

We interpret  $[\ell] \times [m] \times [n]$  as an rc poset by drawing each layer  $\{i\} \times [m] \times [n]$  for  $1 \leq i \leq \ell$  as an rc poset and then letting  $\pi(i, j, k) = (i - j + k, i + j + k)$ . For example, see Figure 9.

As usual, we immediately obtain the following corollary of Theorem 5.2.

**Corollary 7.6.** *There is an equivariant bijection between  $J([\ell] \times [m] \times [n])$  under  $P$  and  $J([\ell] \times [m] \times [n])$  under  $\rho$ .*

Figure 10 displays an orbit of  $J([2] \times [3] \times [4])$  under promotion (drawn using code written by J. S. Kim for TikZ).

We now extend the boundary paths of Definition 4.11 to  $J([\ell] \times [m] \times [n])$ .

**Definition 7.7.** Let  $J \in J([\ell] \times [m] \times [n])$ . We define its *boundary path matrix*  $B(J) = \{b_{i,j}\}$  to be the  $\ell \times (m + n + \ell - 1)$  matrix with row  $i$  containing the boundary path of layer  $\{i\} \times [m] \times [n]$ , preceded by  $i - 1$  zeros and succeeded by  $\ell - i$  zeros.

Note that the rows of the boundary path matrix each sum to  $n$ . Because of the covering relations of  $[\ell] \times [m] \times [n]$ , they also satisfy the condition:

$$\text{if } \sum_{j=1}^k b_{i,j} = \sum_{j=1}^k b_{i+1,j}, \text{ then } b_{i+1,j+1} \neq 1.$$

From our characterization of  $\rho$  as an action on the columns of an rc poset, it is clear that promotion traces from left to right through the columns of the boundary path matrix, swapping each pair of entries in adjacent columns and the same row that result in a matrix still satisfying the condition above. Figure 11 translates the order ideals of Figure 10 to boundary path matrices.

**Theorem 7.8.** *There is an equivariant bijection between  $J([2] \times [m] \times [n])$  under  $P$  and noncrossing partitions of  $[n + m + 1]$  into  $m + 1$  blocks under rotation.*

*Proof.* By Theorem 5.2, we may consider  $J([2] \times [m] \times [n])$  under  $\rho$ .

We first convert a boundary path matrix to a balanced word in  $\beta_{m,n}$  using the correspondence below on columns of the boundary path matrix.

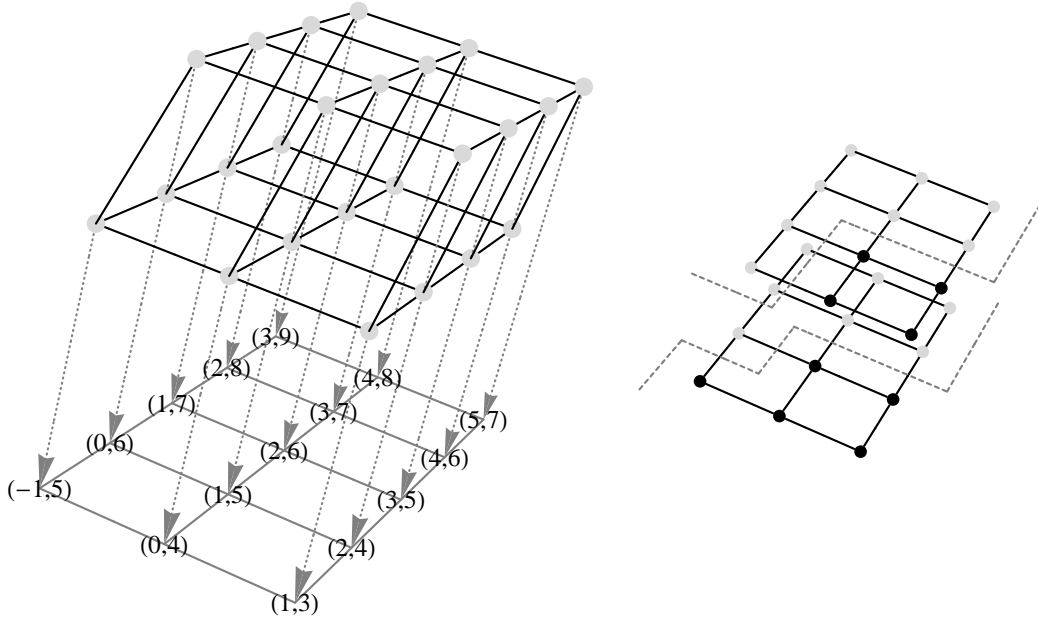


FIGURE 9. On the left is  $[2] \times [3] \times [4]$  drawn as an rc poset of height 2. When there are two elements with the same position, the second element is raised; the position is indicated by a dotted arrow down. Covering relations are drawn with solid black lines, and are projected down as solid gray lines. On the right are the order ideal and boundary paths corresponding to the rightmost plane partition in Figure 10 (covering relations between layers are suppressed).

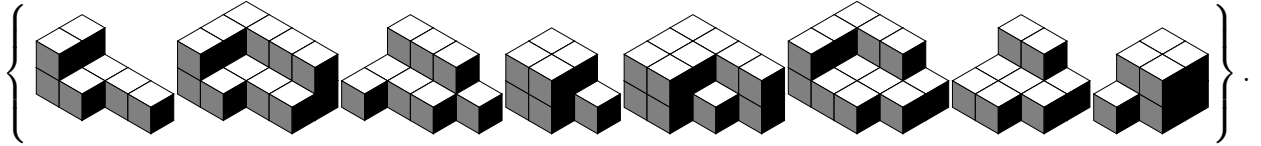


FIGURE 10. An orbit of  $J([2] \times [3] \times [4])$  under promotion.

$$\left\{ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \right. \\ \left. \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \right\}.$$

FIGURE 11. The boundary path matrices of the order ideals in Figure 10.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow '(', \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow ')', \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftrightarrow \boxed{\rangle \langle}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leftrightarrow \bullet$$

Note that the boundary path matrix condition given after Definition 7.7 is equivalent to saying that the resulting words are balanced. Figure 12 translates the boundary path matrices of Figure 11 to balanced words.

We show that this bijection is equivariant, using the definition of  $\psi$  in Theorem 7.5.

The first rule,  $\psi[\bullet A] = A\bullet$ , corresponds to the case when the first column of the boundary path matrix is  $(0,0)^\top$ . This column can swap with all other columns without violating the boundary path matrix condition, and so it is moved to the end of the word under promotion.

Consider when the first column is  $(1,0)^\top$ . This column can swap with  $(0,0)^\top$  and  $(1,0)^\top$  without violating the boundary path matrix condition, but it cannot swap with  $(0,1)^\top$  or  $(1,1)^\top$ .

The second rule,  $\psi[(A_1)A_2] = A_1(A_2)$ , corresponds to when the first column is  $(1,0)^\top$  and the first column it encounters that it cannot swap with is  $(0,1)^\top$ . In this case, the  $(1,0)^\top$  remains fixed, and the  $(0,1)^\top$  is free to move to the end of the word.

The third rule,  $\psi[(A_1)(A_2)(\dots)(A_k)A_{k+1}] = A_1(A_2)(\dots)(A_k)(A_{k+1})$ , corresponds to when the first column  $(1,0)^\top$  encounters  $(1,1)^\top$  first. Then the  $(1,0)^\top$  remains and the  $(1,1)^\top$  can swap to the right without violating the boundary path matrix condition until it reaches the first  $(0,1)^\top$  such that the columns to the left have the same number of 1s in the top and bottom rows. This  $(0,1)^\top$  then continues to the end of the word.

We now give an equivariant bijection from  $\beta_{m,n}$  under  $\psi$  to noncrossing partitions of  $[n+m+1]$  into  $m+1$  blocks under rotation. For  $i < j$ , if ‘(’ in position  $i$  is paired with ‘)’ in position  $j$ —including brackets from the symbol  $(\ )$ —then  $i$  and  $j$  are in a block together. The resulting noncrossing partition will have exactly  $m+1$  blocks because there are  $m+1$  0’s in the bottom row of the boundary path matrix: each  $(0,0)^\top$  column is replaced by a  $\bullet$ , which corresponds to a singleton block, and each  $(1,0)^\top$  column becomes a ‘(’, which corresponds to the first element in a block. For an example, see Figure 12. It is clear that this bijection is equivariant.  $\square$

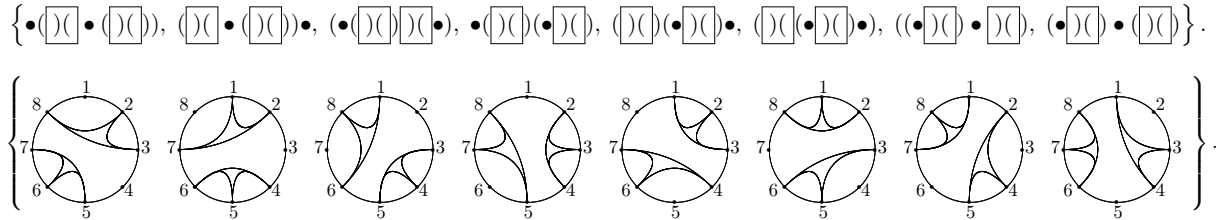


FIGURE 12. The balanced words coming from the boundary path matrices in Figure 11 and the corresponding noncrossing partitions.

**Corollary 7.9** (D. Rush, X. Shi). *Let  $C_{m+n+1}$  act on  $J([2] \times [m] \times [n])$  by  $P$ . Then  $(J([2] \times [m] \times [n]), M_{2,n,m}(q), C_{m+n+1})$  exhibits the CSP.*

*Proof.*  $J([2] \times [m] \times [n])$  under  $P$  is in equivariant bijection with noncrossing partitions of  $[n+m+1]$  into  $m+1$  blocks under rotation, which is known to exhibit the CSP [19].  $\square$

We believe that this bijection can be extended to  $[\ell] \times [m] \times [n]$  for  $\ell > 2$ ; such a bijection would send an element of  $J([\ell] \times [m] \times [n])$  to a noncrossing combinatorial object with  $\ell + m + n - 1$  external vertices, such that promotion translates to rotation of those vertices.

## 8. ASMs AND TSSCPPs

We apply our methods to the alternating sign matrix and totally symmetric self-complementary plane partition posets. In particular, we give an equivariant bijection between ASMs under row-motion and ASMs under B. Wieland's gyration, and we define two actions with related orders on ASMs and TSSCPPs.

### 8.1. The ASM Poset.

**Definition 8.1.** An *alternating sign matrix* (ASM) of order  $n$  is an  $n \times n$  matrix with entries 0, 1, or  $-1$  whose rows and columns sum to 1 and whose nonzero entries in each row and column alternate in sign.

Figure 13 gives the  $3 \times 3$  ASMs.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

FIGURE 13. The seven  $3 \times 3$  ASMs.

ASMs have been objects of much study over the nearly three decades since W. Mills, D. Robbins, and H. Rumsey conjectured [16] that the total number of  $n \times n$  ASMs is

$$(8.1) \quad \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

This conjecture was proved 13 years later independently—and by vastly different methods—by D. Zeilberger [29] and G. Kuperberg [13], and many new developments and directions have emerged since. Nevertheless, an outstanding open problem is to find an explicit bijection between  $n \times n$  ASMs and either of two sets of combinatorial objects known to be equinumerous with them: totally symmetric self-complementary plane partitions (TSSCPPs) inside a  $2n \times 2n \times 2n$  box and descending plane partitions with largest part at most  $n$ .

We begin by recalling the poset interpretation of ASMs, first introduced by A. Lascoux and M. Schützenberger in [14]. This poset is usually defined using monotone triangles, but we choose to define it equivalently using height functions because of their connection with gyration in Section 8.2. For many more interpretations of ASMs, see [18].

**Definition 8.2.** A *height function* of order  $n$  is an  $(n+1) \times (n+1)$  matrix with first row and first column  $0, 1, 2, \dots, n$ , last row and last column  $n, n-1, \dots, 1, 0$ , and such that adjacent entries in any row or column differ by 1.

The height functions of order 3 are given in Figure 14.

**Proposition 8.3** ([9]). *A bijection between  $n \times n$  ASMs and height functions of order  $n$  is given by mapping an ASM  $(a_{ij})_{1 \leq i, j \leq n}$  to the height function  $\left(i + j - 2 \left(\sum_{i'=1}^i \sum_{j'=1}^j a_{i'j'}\right)\right)_{0 \leq i, j \leq n}$ .*

Height functions of order  $n$  have a partial ordering given by componentwise comparison of entries. This partially ordered set is a distributive lattice whose poset of join irreducibles we call  $\nabla \mathbf{A}_n$ , so that  $J(\nabla \mathbf{A}_n)$  is in bijection with the set of  $n \times n$  ASMs [14]. See [26] for further discussion.

For convenience—and in analogy with the construction of TSSCPPs in Section 8.4—we construct  $\nabla \mathbf{A}_n$  directly.



$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

FIGURE 14. The seven height functions of order 3.

**Definition 8.4.** For  $n \geq 1$ , let

$$\mathbf{Q} = \mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{Q}_n$$

be a sequence of posets with inclusion maps

$$\alpha_i : \mathcal{Q}_i \rightarrow \mathcal{Q}_{i+1}.$$

Define  $\nabla \mathbf{Q}$  to be the poset with elements from  $\mathcal{Q}_1, \dots, \mathcal{Q}_n$  and their covering relations, along with the additional covering relations that  $x \in \mathcal{Q}_i$  also covers the elements in  $\mathcal{Q}_{i+1}$  that are covered by  $\alpha_i(x)$ .

$J(\nabla \mathbf{Q})$  inherits actions from the toggle groups  $T(\mathcal{Q}_i)$  in at least two useful ways.

**Definition 8.5.** Let

$$\mathbf{T} = T_1, T_2, \dots, T_n,$$

where each  $T_i = \prod_{j=1}^{k_i} t_{q_{i,j}}$  is an element of the toggle group  $T(\mathcal{Q}_i)$ . Let  $k = \max\{k_1, k_2, \dots, k_n\}$ , and let  $t_{q_{i,j}}$  act as an element of  $T(\nabla \mathbf{Q})$  using the obvious inclusion of  $\mathcal{Q}_i$  in  $\nabla \mathbf{Q}$ . Define

$$(1) \nabla \mathbf{T} = T_1 T_2 \cdots T_n, \text{ and}$$

$$(2) \bar{\nabla} \mathbf{T} = \prod_{j=k}^1 \prod_{i=1}^n t_{q_{i,k_i-j}}, \text{ where we take } t_{q_{i,j}} \text{ to be 1 if } j < 1.$$

In words,  $\nabla \mathbf{T}$  acts by  $T_i$  on  $\mathcal{Q}_i$  in  $\nabla \mathbf{Q}$  before moving on to  $\mathcal{Q}_{i-1}$ . On the other hand,  $\bar{\nabla} \mathbf{T}$  cycles through the  $\mathcal{Q}_i$  in  $\nabla \mathbf{Q}$ , performing one toggle from each  $T_i$  at a time.

**Example 8.6.** Let  $\mathbf{Q} = \mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{Q}_n$  be defined by taking  $\mathcal{Q}_i$  to be the chain with  $i$  elements  $1, \dots, i$ , under the ordering  $1 > \cdots > i$ . If  $j \in \mathcal{Q}_i$ , define  $\alpha_i(j) = j \in \mathcal{Q}_{i+1}$ . Then  $\nabla \mathbf{Q}$  is the dual of the positive root poset  $\Phi^+(A_n)$ . This is the reason for our notation  $\nabla$ .

If we further let  $T_i = t_1 t_2 \cdots t_i$  for each  $\mathcal{Q}_i$ , then  $\nabla \mathbf{T}$  acts as promotion and  $\bar{\nabla} \mathbf{T}$  acts as rowmotion.

**Proposition 8.7.** Let

$$\mathbf{A}_n = \Phi^+(A_1) \hookrightarrow \Phi^+(A_2) \hookrightarrow \cdots \hookrightarrow \Phi^+(A_{n-1}),$$

with  $\alpha_i(e_j - e_k) = e_j - e_{k+1}$ . Then  $J(\nabla \mathbf{A}_n)$  is in bijection with the set of  $n \times n$  ASMs.

Figure 15 gives the order ideals of  $\nabla \mathbf{A}_3$ . Note that one could perform a similar construction with other positive root posets. For example, the same construction with  $\Phi^+(B_i)$  gives a poset whose order ideals are in bijection with  $n \times n$  ASMs that are symmetric about a diagonal.

We can draw  $\nabla \mathbf{A}_n$  as an rc poset of height  $h = n - 1$  by sending  $e_i - e_j$  in  $\Phi^+(A_k)$  to the position  $(i + j + n - k, i - j + n - k)$ . See Figure 16.

Since  $\nabla \mathbf{A}_n$  is an rc poset, Theorem 5.2 applies, giving an equivariant bijection between promotion and rowmotion on the ASM poset.

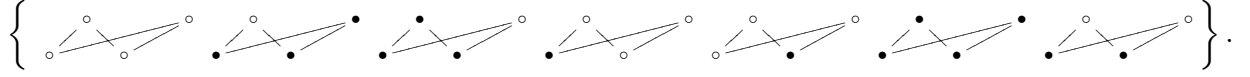


FIGURE 15. The seven order ideals in  $J(\nabla \mathbf{A}_3)$ . They form a single orbit under superpromotion (Definition 8.16).

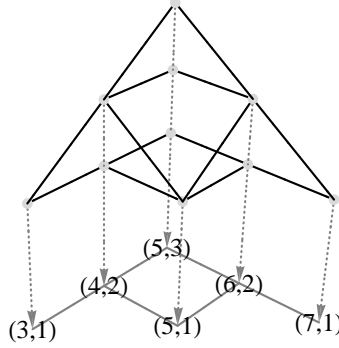


FIGURE 16.  $\nabla \mathbf{A}_4$  drawn as an rc poset of height 3. When there are multiple elements with the same position, subsequent elements are raised; the position is indicated by a dotted arrow down. Covering relations are drawn with solid black lines, and are projected down as solid gray lines.

**Corollary 8.8.** *There is an equivariant bijection between  $J(\nabla \mathbf{A}_n)$  under  $P$  and under  $\rho$ .*

If  $T_i$  is rowmotion on  $J(\Phi^+(A_i))$ , then  $\overline{\nabla} \mathbf{T}$  acts as rowmotion on  $J(\nabla \mathbf{A}_n)$ .

When we let  $T_i$  be promotion on  $J(\Phi^+(A_i))$ , then  $\nabla \mathbf{T}$  does not equal promotion on  $J(\nabla \mathbf{A}_n)$ —we consider this action in Section 8.3.

Interestingly, a conjugate to rowmotion and promotion in the toggle group of  $\nabla \mathbf{A}_n$  has already been studied.

## 8.2. Rowmotion and Gyration.

**Definition 8.9.** Consider the grid  $[n] \times [n]$ . A *fully-packed loop configuration (FPL)* of order  $n$  is a set of paths that begin and end only at every second outward-pointing edge, such that each of the  $n^2$  vertices within the grid lie on exactly one path.

Figure 17 gives the FPLs of order 3.

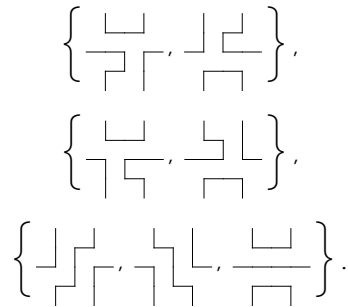


FIGURE 17. The seven FPLs of order 3. They break into three orbits under gyration.

We recall from Proposition 8.3 that height functions of order  $n$  are in bijection with  $n \times n$  ASMs. Thus, FPLs can be seen to be in bijection with ASMs through height functions via the following.

**Proposition 8.10** ([9]). *Height functions of order  $n$  are in bijection with FPLs of order  $n$ .*

*Proof.* We sketch one direction of the bijection (see [9]).

Draw directed edges between adjacent entries in the height function matrix, pointing from the smaller value to the larger value. Rotate each of these edges a quarter-turn counterclockwise about its midpoint and label each vertex even or odd according to the parity of the sum of its row and column indices. Now delete all edges that exit odd vertices and enter even vertices, and undirect the remaining edges.  $\square$

**Definition 8.11.** Pairing up the boundary edges of each path reduces the FPL to a noncrossing matching on  $2n$  vertices. This matching is called the *link pattern* of the FPL.

In 2000, B. Wieland defined an action called *gyration* on FPLs, which he proved rotated the corresponding link pattern [27]. This resolved the refinement by H. Cohn and J. Propp of a conjecture of C. Bosley and L. Fidkowski that the number of FPLs with a certain link pattern depends only on the link pattern. Similar actions had been studied by W. Mills, D. Robbins, and H. Rumsey in [16], though without the combinatorial significance of B. Wieland's result.

In 2010, L. Cantini and A. Sportiello generalized gyration in their proof of the Razumov-Stroganov conjecture that the number of FPLs with a given link pattern appear as the groundstate components of the  $O(1)$  loop model of statistical physics [6].

**Definition 8.12.** Given an FPL, its *gyration* is computed by first visiting all squares with lower left-hand corner  $(i, j)$  for which  $i + j$  is even, and then all squares for which  $i + j$  is odd, swapping the edges around a square if the edges are parallel and otherwise leaving them fixed.

Figure 17 lists the FPLs by orbits under gyration. We can define gyration directly on height functions.

**Proposition 8.13.** *Gyration acts on height functions by going through all the entries  $(i, j)$ , first those for which  $i + j$  is even and then odd, toggling the entry to the other possible value if each adjacent entry is equal and leaving it fixed otherwise.*

Using this interpretation of gyration on height functions, we may interpret gyration directly in terms of the toggle group of the poset  $\nabla \mathbf{A}_n$ .

**Proposition 8.14.** *Gyration acts as  $P_{135\dots 246\dots}$  on  $J(\nabla \mathbf{A}_n)$ .*

*Proof.* The interior height function entries on a diagonal with  $i + j$  of fixed parity correspond to poset elements in rows of the opposite parity, so gyration moves through the poset toggling elements in odd rows first, then those in even rows. This corresponds to  $P_{135\dots 246\dots}$  on  $J(\nabla \mathbf{A}_n)$ .  $\square$

Therefore, by Lemma 5.1, we conclude that rowmotion and gyration are conjugate elements.

**Theorem 8.15.** *There is an equivariant bijection between  $J(\nabla \mathbf{A}_n)$  under rowmotion and under gyration.*

**8.3. ASM Superpromotion.** Though gyration rotates FPL link patterns with order  $2n$ , on the FPLs themselves, gyration has order greater than  $2n$  for  $n > 4$ . Though the order of gyration does not seem to adhere to a simple pattern, we can conclude that it is always divisible by  $2n$  since for each  $n$  there is a link pattern with order exactly  $2n$ . For example, the order of gyration (and rowmotion/promotion) on  $\nabla \mathbf{A}_n$  for  $n = 1, 2, 3, 4, 5, 6, 7$  is  $1, 2, 6, 8, 20, 2520, 3686760$ . Using the construction from Section 8.1, we define an action on ASMs that has order always divisible by  $3n - 2$ .

**Definition 8.16.** Let

$$\mathbf{T} = \rho, \rho, \dots, \rho,$$

where each  $\rho$  acts as promotion on  $\Phi^+(A_i)$ . Define *ASM superpromotion* on  $\nabla \mathbf{A}_n$  by  $\varrho = \nabla \mathbf{T}$ .

Figure 15 lists the single cycle of  $J(\nabla \mathbf{A}_3)$  under  $\varrho$ .

**Theorem 8.17.**  $J(\nabla \mathbf{A}_n)$  under  $\varrho$  has order divisible by  $3n - 2$ .

*Proof.* We show that  $\varrho^{3n-2}$  applied to the empty order ideal is the identity.

Applying  $\varrho^{n-1}$  to the empty order ideal gives the order ideal such that the restriction to the  $\Phi^+(A_k)$  in  $\nabla \mathbf{A}_n$  is the order ideal containing the elements  $e_i - e_j$ , for  $i < j \leq 2k - n + 1$ . Applying  $\varrho^n$  to this order ideal then gives the order ideal containing all elements of  $\nabla \mathbf{A}_n$ . Finally,  $\varrho^{n-1}$  takes us back to the empty order ideal by removing one row from the order ideal at a time.  $\square$

While this action is of order  $3n - 2$  for  $n \leq 6$ , it has order  $3 \cdot (3 \cdot 7 - 2) = 57$  for  $n = 7$  (see Figure 20).

The obvious  $q$ -analogue of (8.1) is  $\prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}$ , which is the generating function of descending plane partitions (DPPs) with largest part at most  $n$  [16]. In general, a corresponding statistic on ASMs or TSSCPPs is not known. For permutation matrices, however, such a statistic is known, and there is a statistic-preserving bijection to a subclass of DPPs [25].

It is not hard to check that this polynomial cannot exhibit the CSP with superpromotion, though the first time it fails is at  $n = 6$ . CSPs for ASMs acted on by cyclic groups of small order have been shown for quarter-turns and half-turns in [24]. For a related result on vertically symmetric ASMs, see [28].

Much as FPLs demonstrate that the order of gyration is divisible by  $2n$ , we believe that there is a generalization of the map stated in Proposition 4.14 to take ASMs to some noncrossing combinatorial object on  $3n - 2$  external vertices.

**8.4. The TSSCPP Poset.** For our purposes, we need only define the poset whose order ideals are in bijection with totally symmetric self-complementary plane partitions; see [26] for a definition of TSSCPPs and an explanation of how the partial order is obtained (this partial order is the same as the partial order on the *magog* triangles of [29]).

**Definition 8.18.** For  $n \geq 1$ , let

$$\mathbf{Q} = \mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2 \hookrightarrow \dots \hookrightarrow \mathcal{Q}_n$$

be a sequence of posets with inclusion maps

$$\beta_i : \mathcal{Q}_i \rightarrow \mathcal{Q}_{i+1}.$$

Define  $\Delta \mathbf{Q}$  to be the poset with elements from  $\mathcal{Q}_1, \dots, \mathcal{Q}_n$  and their covering relations, along with the additional covering relations that  $x \in \mathcal{Q}_i$  is covered by the element  $\beta_i(x) \in \mathcal{Q}_{i+1}$ .

$J(\Delta \mathbf{Q})$  inherits actions from the toggle groups  $T(\mathcal{Q}_i)$  in the same two ways as  $J(\nabla_{i=m}^n \mathcal{Q}_i)$ .

**Definition 8.19.** Let

$$\mathbf{T} = T_1, T_2, \dots, T_n,$$

where each  $T_i = \prod_{j=1}^{k_i} t_{q_{i,j}}$  is an element of the toggle group  $T(\mathcal{Q}_i)$ . Let  $k = \max\{k_1, k_2, \dots, k_n\}$ , and let  $t_{q_{i,j}}$  act as an element of  $T(\nabla \mathbf{Q})$  using the obvious inclusion of  $\mathcal{Q}_i$  in  $\nabla \mathbf{Q}$ .

(1)  $\Delta \mathbf{T} = T_1 T_2 \dots T_n$ , and

(2)  $\underline{\Delta} \mathbf{T} = \prod_{j=k}^1 \prod_{i=1}^n t_{q_{i,k_i-j}}$ , where we take  $t_{q_{i,j}}$  to be 1 if  $j < 1$ .

As with  $\nabla \mathbf{Q}$ ,  $\Delta \mathbf{T}$  acts by  $T_i$  on  $\mathcal{Q}_i$  in  $\Delta \mathbf{Q}$  before moving on to  $\mathcal{Q}_{i-1}$ , while  $\underline{\Delta} \mathbf{T}$  cycles through the  $\mathcal{Q}_i$  in  $\Delta \mathbf{Q}$ , performing one toggle from each  $T_i$  at a time.

**Example 8.20.** Let  $\mathbf{Q} = \mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2 \hookrightarrow \dots \hookrightarrow \mathcal{Q}_n$  be defined by taking  $\mathcal{Q}_i$  to be the chain with  $i$  elements  $1, \dots, i$ , under the ordering  $1 > \dots > i$ . If  $j \in \mathcal{Q}_i$ , define  $\beta_i(j) = j \in \mathcal{Q}_{i+1}$ . Then  $\Delta \mathbf{Q}$  is the positive root poset for type  $A_n$ . This is the reason for our notation  $\Delta$ .

If we instead take  $\beta_i(j) = j + 1 \in \mathcal{Q}_{i+1}$ , then we obtain the poset  $J([2] \times [n - 1])$ .

**Proposition 8.21.** Let

$$\mathbf{A}_n = \Phi^+(A_1) \hookrightarrow \Phi^+(A_2) \hookrightarrow \dots \hookrightarrow \Phi^+(A_{n-1}),$$

with  $\beta_i(e_j - e_k) = e_j - e_k$ . Then  $J(\Delta \mathbf{A}_n)$  is in bijection with the set of TSSCPPs inside a  $2n \times 2n \times 2n$  box.

Figure 18 gives the order ideals of  $\Delta \mathbf{A}_3$ .

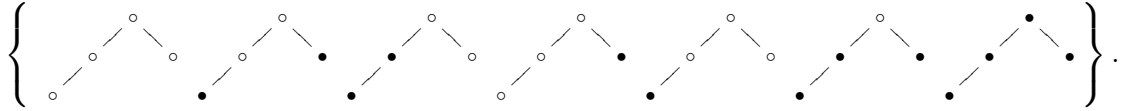


FIGURE 18. There are seven order ideals in  $J(\Delta \mathbf{A}_3)$ . They form a single orbit under rowmotion.

We can draw  $\Delta \mathbf{A}_n$  as an rc poset of height  $\lfloor \frac{n}{2} \rfloor$  by letting

We can draw  $\nabla \mathbf{A}_n$  as an rc poset of height  $h = \lfloor \frac{n}{2} \rfloor$  by sending  $e_i - e_j$  in  $\Phi^+(A_k)$  to the position  $(i + j + k, j - i + k)$ . See Figure 19.

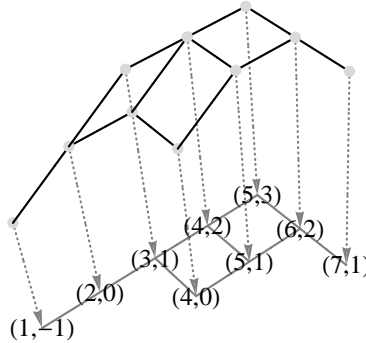


FIGURE 19.  $\Delta \mathbf{A}_4$  drawn as an rc poset of height 2. When there are two elements with the same position, the second element is raised; the position is indicated by a dotted arrow down. Covering relations are drawn with solid black lines, and are projected down as solid gray lines.

Appealing once again to Theorem 5.2, we obtain the conjugacy of  $P$  and  $\rho$ .

**Corollary 8.22.** *There is an equivariant bijection between  $J(\Delta \mathbf{A}_n)$  under  $P$  and under  $\rho$ .*

**Theorem 8.23.**  *$J(\Delta \mathbf{A}_n)$  under  $P$  has order divisible by  $3n - 2$ .*

*Proof.* We show that  $P^{3n-2}$  applied to the empty order ideal is the identity.

Applying  $P$  to the empty order ideal gives the full order ideal. Applying  $P^{n-1}$  to this gives the order ideal such that the restriction to the  $\Phi^+(A_k)$  in  $\Delta \mathbf{A}_n$  is the order ideal containing the elements  $e_i - e_j$ , for  $i < j$  and  $j - i \leq n - 1 - k$ . Applying  $P^{n-1}$  again returns this same order ideal, with the additional elements  $e_i - e_j$  with  $i$  even and  $j - i = n - k$  in  $\Phi^+(A_k)$  (for  $k > \frac{n}{2}$ ). Finally,  $P^{n-1}$  takes us back to the empty order ideal.  $\square$

In analogy with FPLs and ASMs, we again expect a bijection from TSSCPPs to a noncrossing combinatorial object with  $3n - 2$  external vertices, such that promotion translates to rotation of those vertices.

Since the order of  $\rho$  or  $P$  on  $J(\Delta \mathbf{A}_n)$  and the order of  $\varrho$  on  $J(\nabla \mathbf{A}_n)$  are related, one could hope to define a bijection from ASMs to TSSCPPs inductively by associating elements of  $J(\nabla \mathbf{A}_n)$  that are naturally elements of  $J(\nabla \mathbf{A}_{n-1})$ , and then using the orbit structure to extend the bijection. Unlike the situation for positive root posets [1], however, it is not the case that every orbit contains such an element. This method therefore fails—though for small  $n$  it can be used to gather data when trying to find a bijection.

	$J(\nabla \mathbf{A}_n)$ under $\varrho$		$J(\Delta \mathbf{A}_n)$ under $\rho$ or $P$	
	Orbit Size	Number of Orbits	Orbit Size	Number of Orbits
$n = 1$	1	1	1	1
$n = 2$	2	1	2	1
$n = 3$	7	1	7	1
$n = 4$	10	3	10	3
	5	2	5	2
	2	1	2	1
$n = 5$			39	1
			26	1
	13	33	13	28
$n = 6$			112	1
			96	2
			80	2
			64	5
			48	23
			32	30
			24	2
	16	456	16	277
	8	16	8	13
	4	2		
	2	2	2	2
$n = 7$	57	55	?	?
	19	11327	?	?

FIGURE 20. Data for the orbits of  $J(\nabla \mathbf{A}_n)$  under superpromotion and the orbits of  $J(\Delta \mathbf{A}_n)$  under promotion or rowmotion.

## 9. ACKNOWLEDGMENTS

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